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THE CONTROL OF AN ELASTIC MANIPULATOR TAKING INTO ACCOUNT THE USEFUL LOAD AND THE FORCE OF GRAVITY†

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The problem of controlling the motion of a manipulator, consisting of an elastic beam and a useful load, fixed at one of its end, is considered. The arm of the manipulator can rotate in a horizontal plane and move in a vertical direction. It is assumed that a gravitational force acts on the system. It is required to transfer the manipulator from a given initial position to a final one without exciting oscillations. Elastic tension and flexural deformations are taken into account. The control is constructed in the form of series in powers of a parameter, inversely proportional to Young's modulus. Recurrence formulae are given for all the expansion coefficients.

In some problems of the construction and development of robotic systems it is required to take into account the elastic compliance of their components. As a rule, the influence of the gravitational force is ignored when investigating the controlled motion of manipulators with elastic sections [1–6]. Plane rotating motions of an elastic inextensible rod loaded by an absolutely rigid body under the action of the controlling moment of force were analysed and control problems for the transfer of the system from some initial position to a specified angular one with damping of elastic oscillations or to a rotational state of the system as a whole with a fixed angular velocity were considered in [1]. Damping of the oscillations of a massive load, fixed at the end of a long elastic beam by means of an active vibration damper with a translationally moving mass was studied in [2]. A system of partial integro-differential equations and boundary conditions for a manipulator, which consists of a rigid body and an elastic inextensible rod, was obtained taking the effect of the gravitational force into account in [3]. Control problems of transferring the manipulator from an arbitrary initial state to a final one with damping of relative deviations in a time T were investigated. A scheme for solving this problem by reducing it to a certain problem of mathematical physics and a moment problem was suggested, assuming the centrifugal forces to be negligibly small. The controlled motion of an elastic rod taking into account the tensional and flexural deformations was analysed in [4]. The problem of transferring the rod from a specified initial state to a specified final angular one without exciting oscillations when the angular velocity and elastic deviations at the beginning and end of the motion are equal to zero was solved. The control of manipulators with elastic sections and a useful load was studied in [5]. The problem of transferring the manipulator from a specified initial state to a specified final one without exciting oscillations was solved. The control was constructed in the form of a series in powers of a parameter inversely proportional to Young's modulus. A method of solving the problem of the time-optimal action for a simplified model of the manipulator with elastic sections was suggested in [6].

In this paper a method of investigating the controlled motion of a manipulator consisting of an elastic beam and a useful load, fixed at the one of its ends, acted upon by gravitation forces, taking into account torsion and tensional and flexural deformations is developed. The problem of transferring the useful load from a specified initial state to a specified final one is solved. The motion is divided into two stages: (1) the transfer of the manipulator to the desired state; (2) imparting the desired angular orientation

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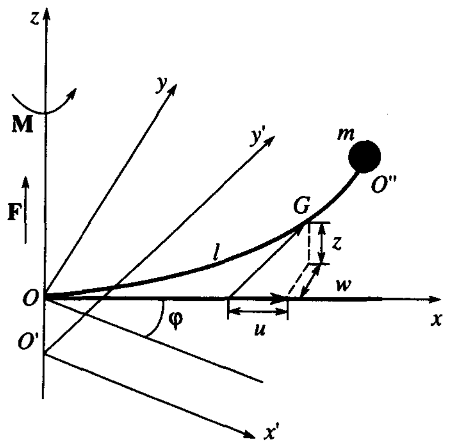

Fig. 1

to the load. The scheme of analysis proposed is based on the semi-inverse method of solving dynamic problems, and it enables the law of variation of the instantaneous form of a beam to be obtained and enables the control moment to be constructed with a previously specified occurring of the expansion of the solution in powers of a parameter that is inversely proportional to Young's modulus.

1. THE EQUATIONS OF MOTION OF A LOADED BEAM

The system in question consists of an elastic uniform beam and a useful load, fixed at one of its ends. The beam is a long uniform bar of arbitrary cross-section area with the neutral line passing through the centre of mass of the cross-section. The neutral line is assumed to be straight in the undeformed state. The beam can rotate in space $O'x'y'z$ about the axis $O'z$ under the control moment **M**. The point O of the beam can move along the $O'z$ axis under the control force **F**. Let l be the length of the beam, ϱ its density, σ its cross-section area, m the mass of the load and g the acceleration due to gravity. We will assume that displacements of points of the beam are small compared to its length. We will introduce a system of coordinates $Oxyz$, which rotates about the $O'z$ axis together with the beam. The Ox axis is specified in a direction tangential to the neutral line of the beam at the point O (Fig. 1).

Let x be the distance from the end O of the undeformed beam to some point G on the neutral line. Suppose $\delta\mathbf{r}$ is the vector of the displacement of the point G, while $u(t, x)$, $w(t, x)$ and $z(t, x)$ are the projections of the vector $\delta\mathbf{r}$ onto the axes of system of coordinates $Oxyz$. The displacement $\delta\mathbf{r}(t, x)$ is a vector function of argument x that determines the instantaneous form of the beam at every fixed instant of time. For a fixed value of x, the displacement $\delta\mathbf{r}(t, x)$ is a vector function of time, which uniquely defines the position of the corresponding point of the neutral line of the beam.

Suppose φ is the angle of rotation of the frame of reference $Oxyz$, which is measured from the axis passing through the point O parallel to the $O'x'$ axis and $(r_1, r_2, r_3)^T$ are the coordinates of the centre of mass of the load in the system of coordinates $O''x''y''z''$ linked with it. The origin of this system is at the fixed point O'' of the load and the beam (the $O''x''$ axis is directed along the tangent to the neutral line of the beam at the point O''; if $\varphi = 0$ and the beam is not deformed, the axes $O''x''$, $O''y''$ and $O''z''$ are parallel to the axes $O'x'$, $O'y'$ and $O'z$, respectively). We will assume that r_1, r_2 and r_3 are small compared to the beam's length. Let $O''x''$, $O''y''$ and $O''z''$ be the principal axes of inertia of the load, and let A, B and C be the principal moments of inertia of the load about the axes $O''z''$, $O''y''$ and $O''x''$, respectively.

We will connect the system of coordinates $Gx'''y'''z'''$ to each cross-section of the beam, and direct the Gx''' axis along the tangent to the neutral line of the beam at the point G. Let γ be the angle of rotation of the cross-section of the beam about the Gx''' axis. Taking into account the rotational inertia of the load and the cross-section of the beam [7], the kinetic energy of the system can be represented in the form

$$T = \frac{1}{2}\int_0^l \rho\sigma\mathcal{F}(u, u_t, w, w_t, z_t, \tilde{z}_t, x, \dot{\varphi})dx + \frac{1}{2}m\tilde{\mathcal{F}}(a, a_t, b, b_t, c_t, l, \dot{\varphi}) + T_1$$

$$a(t) = u(t, l) + r_1 - r_2 w_x(t, l) - r_3 z_x(t, l)$$

$$b(t) = w(t, l) + r_2 + r_1 w_x(t, l) - r_3 \gamma(t, l)$$

$$c(t) = z(t, l) + \tilde{z}(t) + r_3 + r_1 z_x(t, l) + r_2 \gamma(t, l)$$

$$\mathcal{F}(u, u_t, w, w_t, z_t, \tilde{z}_t, x, \dot{\varphi}) = (x\dot{\varphi} + w_t)^2 + w^2\dot{\varphi}^2 + 2xu\dot{\varphi}^2 + u^2\dot{\varphi}^2 +$$

$$+ u_t^2 + 2u\dot{\varphi}w_t - 2w\dot{\varphi}u_t + (z_t + \tilde{z}_t)^2$$

$$\tilde{\mathcal{F}}(a, a_t, b, b_t, c_t, l, \dot{\varphi}) = (l\dot{\varphi} + b_t)^2 + b^2\dot{\varphi}^2 + 2la\dot{\varphi}^2 + a^2\dot{\varphi}^2 + a_t^2 + 2a\dot{\varphi}b_t - 2b\dot{\varphi}a_t + c_t^2$$

$$T_1 = \frac{1}{2}\int_0^l \varrho I_1 (w_{xt} + \dot{\varphi})^2 dx + \frac{1}{2}A(w_{xt}(t, l) + \dot{\varphi})^2 +$$

$$+ \frac{1}{2}\int_0^l \varrho(I_2 z_{tx}^2 + I_3\gamma_t^2)dx + \frac{1}{2}(Bz_{tx}^2(t, l) + C\gamma_t^2(t, l))$$

where $\tilde{z}(t)$ is the absolute vertical coordinate of the point O, and I_1, I_2 and I_3 are the moments of inertia of the cross-section about the axes Gz''', Gy''' and Gx''', respectively.

The potential energy has the form (1.1)

$$\Pi = \frac{1}{2}\int_0^l E(I_1 w_{xx}^2 + I_2 z_{xx}^2)dx + \frac{1}{2}\int_0^l EI_3\gamma_x^2 dx + \int_0^l g\varrho\sigma(z + \tilde{z})dx +$$
$$+ \frac{1}{2}\int_0^l E\sigma\left(\sqrt{(1 + u_x)^2 + w_x^2 + z_x^2} - 1\right)^2 dx + mg(z(t, l) + \tilde{z}) + mg(r_2 \sin\gamma + r_3 \cos\gamma) \tag{1.1}$$

where E is Young's modulus.

The first term on the right-hand side of equality (1.1) is the potential energy of elastic bending deformation, the second term is the torsional potential energy, the third, fifth and sixth terms are the potential energy due to the weight of the beam and the load, and the fourth term is the potential energy of elastic stretching deformation.

We will introduce the dimensionless variables (everywhere below $k = 1, 2, 3$)

$$t' = \frac{t}{\tau}, \quad u' = \frac{u}{l}, \quad w' = \frac{w}{l}, \quad z' = \frac{z}{l}, \quad \tilde{z}' = \frac{\tilde{z}}{l}, \quad x' = \frac{x}{l}, \quad r_k' = \frac{r_k}{l}$$

where τ is the time scale, and

$$\frac{1}{\tau^2} < \frac{EI}{\varrho\sigma l^4} < \frac{E}{\varrho l^2} \tag{1.2}$$

Suppose M_0 and F_0 are characteristic quantities having dimensions of moment and force, respectively, e.g. $M_0 = \sup_t|\mathbf{M(t)}|$ and $F_0 = \sup_t|\mathbf{F(t)}|$. We will assume that u_x, w_x and z_x are small compared to unity. Let the following inequalities hold

$$\dot{\varphi}^2 \ll \frac{1}{\tau^2}, \quad \ddot{\varphi} \ll \frac{1}{\tau^2}$$

The system of equations of motion and the boundary conditions in new variables can be written using the Ostrogradskii–Hamilton's principle [8], taking into account assumptions made, in the form (the primes are omitted) [5]

$$\beta\int_0^1[x^2\ddot{\varphi}+xw_{tt}+2x\dot{\varphi}u_t]dx+\beta\alpha[\ddot{\varphi}+w_{tt}+2\dot{\varphi}u_t]\big|_{x=1}+$$

$$+\beta_1\int_0^1(\ddot{\varphi}+w_{xtt})dx+\beta_2(w_{xtt}(t,1)+\ddot{\varphi}) = Q_1$$

$$\tilde{\beta}\int_0^1 z_{tt}dx+\tilde{\beta}(1+\alpha)\tilde{z}_{tt}+\tilde{\beta}_3+\alpha\tilde{\beta}z_{tt}(t,1)+\int_0^1\tilde{\beta}_1 z_{xtt}dx+\tilde{\beta}_2 z_{xtt}(t,1) = Q_2$$

$$\mu\chi_1\left(x\ddot{\varphi}+w_{tt}-\frac{1}{\chi_1}w_{xxtt}+2\dot{\varphi}u_t\right)+w_{xxxx} = 0$$

$$\mu\chi_2\left(z_{tt}-\frac{1}{\chi_2}z_{xxtt}+\tilde{z}_{tt}+\Delta\right)+z_{xxxx} = 0$$

$$\gamma_{xx} = \mu\gamma_{tt}\mu(x\dot{\varphi}^2-u_{tt}+2\dot{\varphi}w_t)+u_{xx} = 0$$

$$w_{xx}(t,1) = -\mu\chi_1\lambda_1(w_{xtt}+\ddot{\varphi})\big|_{x=1}$$

$$z_{xx}(t,1) = -\mu\chi_2\lambda_2 z_{xtt}\big|_{x=1} \tag{1.3}$$

$$w_{xxx}(t,1) = \mu\chi_1\alpha[\ddot{\varphi}+w_{tt}+2\dot{\varphi}u_t]\big|_{x=1}+\mu(w_{xtt}+\ddot{\varphi})\big|_{x=1}$$

$$z_{xxx}(t,1) = \mu\chi_2\alpha[\tilde{z}_{tt}+z_{tt}+\Delta]\big|_{x=1}+\mu z_{xtt}\big|_{x=1}$$

$$\gamma_x(t,1) = -\mu\lambda_3\gamma_{tt}(t,1)-\mu\alpha\chi_3(r_2\Delta-r_3\ddot{\varphi}+r_2\tilde{z}_{tt})$$

$$u_x(t,1) = \mu\alpha[\dot{\varphi}^2-u_{tt}+2\dot{\varphi}w_t]\big|_{x=1}$$

$$L = \frac{m}{\rho\sigma},\quad \alpha = \frac{L}{l},\quad \beta = \frac{\rho\sigma l^3}{\tau^2 M_0},\quad \beta_1 = \frac{\rho I l}{\tau^2 M_0},\quad \beta_2 = \frac{A}{\tau^2 M_0}$$

$$\tilde{\beta} = \frac{\rho\sigma l^2}{\tau^2 F_0},\quad \tilde{\beta}_1 = \frac{\rho l I_3}{\tau^2 F_0},\quad \tilde{\beta}_2 = \frac{B}{\tau^2 F_0},\quad \tilde{\beta}_3 = \frac{g}{F_0}(m+\rho\sigma l)$$

$$Q_1 = \frac{M}{M_0},\quad Q_2 = \frac{F}{F_0},\quad \mu = \frac{\rho l^2}{E\tau^2},\quad \chi_k = \frac{\sigma l^2}{I_k}$$

$$\lambda_1 = \frac{A}{\rho\sigma l^3},\quad \lambda_2 = \frac{B}{\rho\sigma l^3},\quad \lambda_3 = \frac{C}{\rho\sigma l^3},\quad \Delta = \frac{g\tau^2}{l}$$

Since the end of the beam O is fixed and restrained, it is necessary to add six more conditions to the above equations:

$$\begin{aligned} &w(t,0) = 0,\quad w_x(t,0) = 0,\quad u(t,0) = 0\\ &z(t,0) = 0,\quad z_x(t,0) = 0,\quad \gamma(t,0) = 0 \end{aligned} \tag{1.4}$$

The problem of transferring the beam from a specified initial position to a specified final one was solved for the above mathematical model, where the following conditions must be satisfied at the beginning and end of the manoeuvre

$$\varphi(0) = 0,\quad \dot{\varphi}(0) = 0,\quad w(0,x) = w_t(0,x) = u(0,x) = u_t(0,x) = 0$$

$$\tilde{z}(0) = \tilde{z}_t(0) = 0, \quad z(0,x) = Z_0(x), \quad z_t(0,x) = 0, \quad \gamma(0,x) = G_0(x)$$

$$\varphi(\eta) = \varphi_0, \quad \dot{\varphi}(\eta) = 0, \quad w(\eta,x) = w_t(\eta,x) = u(\eta,x) = u_t(\eta,x) = 0 \tag{1.5}$$

$$\tilde{z}(\eta) = z_0, \quad \tilde{z}_t(\eta) = 0$$

$$z(\eta,x) = Z_0(x), \quad z_t(\eta,x) = 0, \quad \gamma(\eta,x) = G_0(x)$$

where

$$\eta = \frac{T}{\tau}, \quad Z_0(x) = -\mu\chi_2\left[\frac{1}{24}x^4 - \frac{1}{6}(\alpha+1)x^3 + \frac{1}{4}(1+2\alpha)x^2\right]\Delta$$

$$G_0(x) = -\mu\chi_3\alpha r_2 x\Delta$$

In reality these conditions mean that the manipulator is transferred from one state of equilibrium to another without exciting oscillations.

2. CONTROL OF AN ELASTIC LOADED BEAM

We will construct the solution of system (1.3), taking conditions (1.4) and (1.5) into account, in the form of series in powers of the parameter μ. Let $\mu = 0$; then

$$u^{(0)}(t,x) \equiv 0, \quad w^{(0)}(t,x) \equiv 0, \quad z^{(0)}(t,x) \equiv 0, \quad \gamma^{(0)}(t,x) \equiv 0$$

Consequently,

$$u = \sum_{i=1}^{\infty}\mu^i u^{(i)}(t,x), \quad w = \sum_{i=1}^{\infty}\mu^i w^{(i)}(t,x)$$
$$z = \sum_{i=1}^{\infty}\mu^i z^{(i)}(t,x), \quad \gamma = \sum_{i=1}^{\infty}\mu^i \gamma^{(i)}(t,x) \tag{2.1}$$

$$\varphi = \sum_{i=0}^{\infty}\mu^i\varphi^{(i)}(t), \quad \tilde{z} = \sum_{i=0}^{\infty}\mu^i\tilde{z}^{(i)}(t), \quad Q_1 = \sum_{i=0}^{\infty}\mu^i Q_1^{(i)}(t), \quad Q_2 = \sum_{i=0}^{\infty}\mu^i Q_2^{(i)}(t) \tag{2.2}$$

Let $\dot{\varphi}^{(0)}(t)$ and $\tilde{z}_t^{(0)}(t)$ be specified functions. We will assume that $\dot{\varphi}^{(i)}(t) \equiv 0$ and $\tilde{z}_t^{(0)}(t) \equiv 0$ for any $i > 0$. We will distinguish the coefficients of the zeroth and first powers of μ in Eqs (1.3) and the boundary conditions. For the first approximation we obtain the following expressions

$$u^{(1)}(t,x) = (\dot{\varphi}^{(0)})^2\frac{x}{2}\left[(2\alpha+1) - \frac{x^2}{3}\right] \tag{2.3}$$

$$w^{(1)}(t,x) = -\frac{\chi_1}{120}\ddot{\varphi}^{(0)}[x^5 - 10(2(\alpha+\chi_1^{-1})+1)x^3 + 20(3(\alpha+\chi_1^{-1}+\lambda_1)+1)x^2] \tag{2.4}$$

$$z^{(1)}(t,x) = -\chi_2\left[\frac{1}{24}x^4 - \frac{1}{6}(\alpha+1)x^3 + \frac{1}{4}(2\alpha+1)x^2\right](\tilde{z}_{tt}^{(0)} + \Delta) \tag{2.5}$$

$$\gamma^{(1)}(t,x) = -\mu\alpha\chi_3(r_2\Delta - r_3\ddot{\varphi}^{(0)} + r_2\tilde{z}_{tt}^{(0)})$$

$$Q_1^{(0)} = \beta\left(\frac{1}{3}+\alpha\right)\ddot{\varphi}^{(0)} + (\beta_1+\beta_2)\ddot{\varphi}^{(0)}$$

$$
\begin{aligned}
Q_1^{(1)}(t) &= \beta\int_0^1 [xw_{tt}^{(1)} + 2x\dot{\varphi}^{(0)}u_t^{(1)}]dx + \beta\alpha[w_{tt}^{(1)} + 2\dot{\varphi}^{(0)}u_t^{(1)}]\big|_{x=1} + \\
&+ \beta_1\int_0^1 w_{xtt}^{(1)}dx + \beta_2 w_{xtt}^{(1)}(t, 1) \\
Q_2^{(0)}(t) &= \tilde{\beta}(1+\alpha)\tilde{z}_{tt}^{(0)} + \tilde{\beta}_3 \\
Q_2^{(1)}(t) &= \tilde{\beta}\int_0^1 z_{tt}^{(1)}dx + \alpha\tilde{\beta}z_{tt}^{(1)}(t, 1) + \int_0^1 \tilde{\beta}_1 z_{xtt}^{(1)}dx + \tilde{\beta}_2 z_{xtt}^{(1)}(t, 1)
\end{aligned}
\tag{2.6}
$$

Separating the coefficients of μ^n, we obtain recurrence formulae for determining the nth approximation $u^{(n)}, w^{(n)}, Q_1^{(n)}, Q_2^{(n)}$ $(n > 1)$

$$
\begin{aligned}
&\beta\int_0^1 [xw_{tt}^{(n)} + 2x\dot{\varphi}^{(0)}u_t^{(n)}]dx + \beta\alpha[w_{tt}^{(n)} + 2\dot{\varphi}^{(0)}u_t^{(n)}]\big|_{x=1} + \\
&+ \beta_1\int_0^1 w_{xtt}^{(n)}dx + \beta_2 w_{xtt}^{(n)}(t, 1) = Q_1^{(n)} \\
&\tilde{\beta}\int_0^1 z_{tt}^{(n)}dx + \alpha\tilde{\beta}z_{tt}^{(n)}(t, 1) + \int_0^1 \tilde{\beta}_1 z_{xtt}^{(n)}dx + \tilde{\beta}_2 z_{xtt}^{(n)}(t, 1) = Q_2^{(n)} \\
&w_{xxxx}^{(n)} = -\chi_1\left[w_{tt}^{(n-1)} - \frac{1}{\chi_1}w_{xxtt}^{(n-1)} + 2\dot{\varphi}^{(0)}u_t^{(n-1)}\right] \\
&z_{xxxx}^{(n-1)} = -\chi_2\left(z_{tt}^{(n-1)} - \frac{1}{\chi_2}z_{xxtt}^{(n-1)}\right) \\
&\gamma_{xx}^{(n)} = \gamma_{tt}^{(n-1)} \\
&u_{xx}^{(n)} = -[-u_{tt}^{(n-1)} + 2\dot{\varphi}^{(0)}w_t^{(n-1)}] \\
&w^{(n)}(t, 0) = w_x^{(n)}(t, 0) \equiv 0 \\
&w_{xx}^{(n)}(t, 1) = -\chi_1\lambda_1 w_{xtt}^{(n-1)}(t, 1) \\
&w_{xxx}^{(n)}(t, 1) = \chi_1\alpha[w_{tt}^{(n-1)} + 2\dot{\varphi}^{(0)}u_t^{(n-1)}]\big|_{x=1} + w_{xtt}^{(n-1)}(t, 1) \\
&z^{(n)}(t, 0) = z_x^{(n)}(t, 0) \equiv 0 \\
&z_{xx}^{(n)}(t, 1) = -\chi_2\lambda_2 z_{xtt}^{(n-1)}(t, 1) \\
&z_{xxx}^{(n)}(t, 1) = \chi_2\alpha z_{tt}^{(n-1)}(t, 1) + z_{xtt}^{(n-1)}(t, 1) \\
&\gamma^{(n)}(t, 0) \equiv 0, \quad \gamma_x^{(n)}(t, 1) \equiv -\lambda_3\gamma_{tt}^{(n-1)}(t, 1) \\
&u^{(n)}(t, 0) \equiv 0 \\
&u_x^{(n)}(t, 1) = \alpha[-u_{tt}^{(n-1)} + 2\dot{\varphi}^{(0)}w_t^{(n-1)}]\big|_{x=1}
\end{aligned}
\tag{2.7}
$$

The last four equations of system (2.7) can be solved, since their right-hand sides are polynomials in powers of x. Then, substituting the expressions obtained for $w^{(n)}$, $z^{(n)}$, $u^{(n)}$ and $\gamma^{(n)}$ into the first and second equations of system (2.7), we obtain the laws of variation of $Q_1^{(n)}(t)$ and $Q_2^{(n)}(t)$.

In order for conditions (1.5) to be satisfied up to the nth order inclusive it is sufficient to choose the functions $\dot{\varphi}^{(0)}(t)$ and $\tilde{z}_t^{(0)}(t)$ so that

$$\begin{aligned}
&\dot{\varphi}^{(0)}(0) = \ddot{\varphi}^{(0)}(0) = \ldots = \frac{d^{2n+1}\varphi^{(0)}}{dt^{2n+1}}(0) = 0 \\
&\tilde{z}_t^{(0)}(0) = \tilde{z}_{tt}^{(0)}(0) = \ldots = \frac{d^{2n+1}\tilde{z}^{(0)}}{dt^{2n+1}}(0) = 0 \\
&\dot{\varphi}^{(0)}(\eta) = \ddot{\varphi}^{(0)}(\eta) = \ldots = \frac{d^{2n+1}\varphi^{(0)}}{dt^{2n+1}}(\eta) = 0 \\
&\tilde{z}_t^{(0)}(\eta) = \tilde{z}_{tt}^{(0)}(\eta) = \ldots = \frac{d^{2n+1}\tilde{z}^{(0)}}{dt^{2n+1}}(\eta) = 0
\end{aligned} \tag{2.8}$$

Conditions (2.8) are obtained from system of equations (2.7) and also from (2.3)–(2.6). The coefficients in expansions (2.1) and (2.2) have a polynomial form with respect to x.

3. THE EQUATIONS OF MOTION OF AN ELASTIC MANIPULATOR WITH A ROTATING USEFUL LOAD

A manipulator consisting of a uniform elastic beam and a load of mass m is considered. The end of the beam O is clamped and restrained. At the point O the tangent to the neutral line of the beam remains fixed. The load is attached to the end of the beam O' and can rotate under the moment $\tilde{\mathbf{M}}$ about the axis coinciding with the tangent at the point O' to the neutral line of the beam. The centre of mass of the load can deviate from the tangent to the neutral line of the beam at the point O'. It is assumed that displacements of points of the beam are small compared to its length.

We will introduce a fixed system of coordinates $Oxyz$. The Ox axis lies along the neutral line of the undeformed beam at the initial instant of time (Fig. 2).

As before, x is the distance from the end O of the undeformed beam to a certain point G on the neutral line. Suppose $\delta\mathbf{r}$ is the displacement vector of the point G and $u(t,x)$, $w(t,x)$ and $z(t,x)$ are the projections of the vector $\delta\mathbf{r}$ onto the axes of the system of coordinates $Oxyz$.

We will relate system of coordinates $Gx'y'z'$ with every cross-section of the beam, where the Gx' axis is directed along the tangent to the neutral line of the beam at the point G. We will represent the kinetic energy of the mechanical system as the sum of two terms: the kinetic energy of the beam T_1 and the kinetic energy of the load T_2

$$T = T_1 + T_2$$

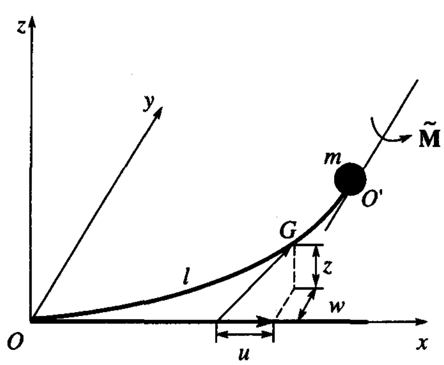

Fig. 2

Suppose γ is a (small) angle of rotation of the cross-section of the beam about the tangent to the neutral line of the beam. We will assume that Gx'. Gy' and Gz' are the principal central axes of inertia of the cross-section of the beam at the point G. We then obtain for the kinetic energy T_1

$$T_1 = \frac{1}{2}\int_0^l \rho\sigma(u_t^2 + w_t^2 + z_t^2)dx + \frac{1}{2}\int_0^l \rho(I_1\dot{\theta}^2 + I_2\dot{\psi}^2 + I_3\dot{\gamma}^2)dx$$

where I_1, I_2, I_3 are the moments of inertia of the cross-section of the beam about the axes Gz', Gy' and Gx', respectively, and θ and ψ are small angles of rotation of the cross-section of the beam about the axes Gz' and Gy', respectively.

The kinetic energy T_2 is calculated from the formula [8]

$$T_2 = \frac{1}{2}m\mathbf{V}_{O'}^2 + m\mathbf{V}_{O'}\cdot(\boldsymbol{\omega}\times\mathbf{r}_c) + \frac{1}{2}\boldsymbol{\omega}\cdot J\boldsymbol{\omega}$$

where $\mathbf{V}_{O'}$ is the velocity of the point O', $\mathbf{r}_c$ is the radius-vector of the centre of mass of the load in system of coordinates $O'x'y''z''$, commented to the load, $\boldsymbol{\omega}$ is the angular velocity and J is the inertia operator.

Suppose the axes $O'x'$, $O'y''$ and $O'z''$ are the principal axes of inertia of the load and the coordinates of the vector $\mathbf{r}_c = (R_1, R_2, R_3)$ are small compared to the length of the beam. We will denote by ϕ the angle of rotation of the load (not a small quantity) about Ox' axis, and by A, B and C the moments of inertia of the load about the principal axes. We will assume that $A/B \sim 1$. Then the kinetic energy of the system can be written in the form [5]

$$T = T_1 + T_2 = \frac{1}{2}\int_0^l \rho\sigma(u_t^2 + w_t^2 + z_t^2)dx + \frac{1}{2}\int_0^l \rho(I_1 w_{xt}^2 + I_2 z_{xt}^2 + I_3\gamma_t^2)dx +$$

$$+\frac{1}{2}m(u_t^2 + w_t^2 + z_t^2)\big|_{x=l} + m\dot{\phi}[-w_t(R_3\cos\phi + R_2\sin\phi) +$$

$$+z_t(-R_3\sin\phi + R_2\cos\phi)]\big|_{x=l} + \frac{1}{2}[B(w_{zt}^2 + z_{xt}^2) + C\dot{\phi}^2]\big|_{x=l}$$

The potential energy is expressed by the relation

$$\Pi = \frac{1}{2}\int_0^l (EI_1 w_{xx}^2 + EI_2 z_{xx}^2)dx + \frac{1}{2}\int_0^l E\sigma\left(\sqrt{(1+u_x)^2 + w_x^2 + z_x^2} - 1\right)^2 dx +$$
$$+\frac{1}{2}\int_0^l EI_3\gamma_x^2 dx + \int_0^l \rho\sigma g z dx + mgz(t,l) + mg(R_2\sin\phi + R_3\cos\phi) \tag{3.1}$$

We will introduce the dimensionless variables

$$t' = \frac{t}{\tau_1},\quad u' = \frac{u}{l},\quad w' = \frac{w}{l},\quad z' = \frac{x}{l},\quad R_k' = \frac{R_k}{l},\quad x' = \frac{x}{l}$$

where τ_1 is a time scale, which satisfies condition (1.2).

Using the Ostrogradskii–Hamilton principle and taking into account the above assumptions, we obtain the equations of motion and the boundary conditions in the new variables (the primes are omitted) [5]

$$\mu u_{tt} - u_{xx} = 0,\quad \mu(\chi_1 w_{tt} - w_{xxtt}) + w_{xxxx} = 0$$

$$\mu(\chi_2 z_{tt} - z_{xxtt} + \chi_2\Delta) + z_{xxxx} = 0,\quad \mu\gamma_{tt} - \gamma_{xx} = 0$$

$$\ddot{\phi} = \tilde{Q} - \Delta_1(R_2\cos\phi - R_3\sin\phi)$$

$$w_{xx}(t,1) + \mu\lambda_1 w_{xtt}(t,1) = 0$$

$$
\begin{aligned}
& z_{xx}(t, 1) + \mu\lambda_2 z_{xtt}(t, 1) = 0 \\
& u_x(t, 1) = -\mu\alpha u_{tt}(t, 1) \\
& \gamma_x(t, l) = -\mu\lambda_3\ddot{\phi} - \mu\alpha\chi_3\Delta(R_2\cos\phi - R_3\sin\phi) \\
& w_{xxx}(t, 1) = \mu\chi_1\alpha[w_{tt}(t, 1) - \ddot{\phi}(R_3\cos\phi + R_2\sin\phi) + \\
& + \dot{\phi}^2(R_3\sin\phi + R_2\cos\phi)] + \mu w_{xtt}(t, 1) \\
& z_{xxx}(t, l) = \mu\chi_2\alpha[z_{tt}(t, l) + \Delta - \ddot{\phi}(R_3\sin\phi - R_2\cos\phi) - \\
& - \dot{\phi}^2(R_3\cos\phi + R_2\sin\phi)] + \mu z_{xtt}(t, 1) \\
& \mu = \frac{\rho l^2}{E\tau_1^2}, \quad \chi_k = \frac{\sigma l^2}{I_k}, \quad \lambda_1 = \frac{B}{\rho l I_1}, \quad \lambda_2 = \frac{B}{\rho l I_2}, \quad \lambda_3 = \frac{C}{\rho l I_3} \\
& \tilde{Q} = \frac{\tilde{M}\tau_1^2}{C}, \quad L = \frac{m}{\rho\sigma}, \quad \alpha = \frac{L}{l}, \quad \Delta = \frac{g\tau^2}{l}, \quad \Delta_1 = \frac{mgl\tau^2}{C}
\end{aligned}
\tag{3.2}
$$

Since the end of the beam O is fixed and restrained, we add the following conditions

$$
w(t, 0) = w_x(t, 0) = w(t, 0) = \gamma(t, 0) = z(t, 0) = z_x(t, 0) = 0
$$

For the above mathematical model the problem of the rotation of the load from the specified initial angular position to the final one was solved, where the following conditions must be satisfied at the beginning and end of the manoeuvre,

$$
\begin{aligned}
& \phi(0) = \phi_0, \quad \dot{\phi}(0) = 0, \quad w(0, x) = w_t(0, x) = 0 \\
& u(0, x) = u_t(0, x) = z_t(0, x) = \gamma_t(0, x) = 0 \\
& z(0, x) = Z_0(x), \quad \gamma(0, x) = G_1(x) \\
& \phi(\eta_1) = \phi_1, \quad \dot{\phi}(\eta_1) = 0, \quad w(\eta_1, x) = w_t(\eta_1, x) = 0 \\
& u(\eta_1, x) = u_t(\eta_1, x) = z_t(\eta_1, x) = 0 \\
& \gamma_t(\eta_1, t) = 0, \quad z(\eta_1, x) = Z_0(x), \quad \gamma(\eta_1, x) = G_2(x)
\end{aligned}
\tag{3.3}
$$

where

$$
\begin{aligned}
& \eta_1 = \frac{T_1}{\tau_1}, \quad Z_0(x) = -\mu\chi_2\left[\frac{1}{24}x^4 - \frac{1}{6}(\alpha + 1)x^3 + \frac{1}{4}(1 + 2\alpha)x^2\right]\Delta \\
& G_1(x) = -\mu\chi_3\alpha\Delta x(R_2\cos\phi_0 - R_3\sin\phi_0) \\
& G_2(x) = -\mu\chi_3\alpha\Delta x(R_2\cos\phi_1 - R_3\sin\phi_1)
\end{aligned}
$$

4. CONTROL OF THE MOTION OF THE LOAD

We will construct the solution of system (3.2) in the form of series in powers of the parameter μ. Suppose $\mu = 0$, then

$$
w^{(0)}(t, x) = u^{(0)}(t, x) = z^{(0)}(t, x) = \gamma^{(0)}(t, x) \equiv 0
$$

Consequently

$$u = \sum_{i=1}^{\infty} \mu^i u^{(i)}(t, x), \quad w = \sum_{i=1}^{\infty} \mu^i w^{(i)}(t, x), \quad z = \sum_{i=1}^{\infty} \mu^i z^{(i)}(t, x) \tag{4.1}$$

$$\gamma = \sum_{i=1}^{\infty} \mu^i \gamma^{(i)}(t, x), \quad \phi = \sum_{i=0}^{\infty} \mu^i \phi^{(i)}(t), \quad \tilde{Q} = \sum_{i=0}^{\infty} \mu^i \tilde{Q}^{(i)}(t) \tag{4.2}$$

Let $\dot{\phi}^{(0)}(t)$ be a specified law of variation of the angular velocity of the load. We will suppose that $\phi^{(i)} \equiv 0$ for any $i > 0$. We will separate the coefficients of the zeroth and first powers of μ in Eqs (3.2) and in the boundary conditions. For the first approximation we obtain the expressions

$$u^{(1)}(t, x) \equiv 0, \quad w^{(1)}(t, x) = f_1(x)\Lambda_1(t), \quad f_1(x) = \frac{1}{2}\chi_1\alpha\left(x^2 - \frac{1}{3}x^3\right) \tag{4.3}$$

$$z^{(1)}(t, x) = -\chi_2\left\{\left[\frac{x^4}{24} - (\alpha+1)\frac{x^3}{6} + \left(\alpha + \frac{1}{2}\right)\frac{x^2}{2}\right]\Delta - \alpha\Lambda_2(t)\left[\frac{x^3}{6} - \frac{x^2}{2}\right]\right\} \tag{4.4}$$

$$\gamma^{(1)}(t, x) = h_1(x)\ddot{\phi}^{(0)} + p_1(x)\Lambda_3(t), \quad h_1(x) = -\lambda_3 x, \quad p_1(x) = -\alpha\chi_3\Delta x \tag{4.5}$$

where

$$\Lambda_1(t) = \ddot{\phi}^{(0)}(R_3\cos\phi^{(0)} + R_2\sin\phi^{(0)}) + (\dot{\phi}^{(0)})^2(R_3\sin\phi^{(0)} + R_2\cos\phi^{(0)})$$

$$\Lambda_2(t) = \ddot{\phi}^{(0)}(R_2\cos\phi^{(0)} - R_3\sin\phi^{(0)}) - (\dot{\phi}^{(0)})^2(R_2\sin\phi^{(0)} + R_3\cos\phi^{(0)})$$

$$\Lambda_3(t) = R_2\cos\phi^{(0)} - R_3\sin\phi^{(0)}$$

$$\tilde{Q}(t) = \ddot{\phi}^{(0)}(t) + \Delta_1\Lambda_3(t)$$

Separating out the coefficients of μ^n, we obtain recurrence formulae for the nth approximation $u^{(n)}$, $w^{(n)}$, $z^{(n)}$, $\gamma^{(n)}$ $(n > 1)$

$$u^{(n)}(t, x) \equiv 0, \quad w^{(n)}_{xxxx} = (f''_{n-1} - \chi_1 f_{n-1})\frac{d^{2(n-1)}}{dt^{2(n-1)}}\Lambda_1(t)$$

$$z^{(n)}_{xxxx} = (g''_{n-1} - \chi_2 g_{n-1})\frac{d^{2(n-1)}}{dt^{2(n-1)}}\Lambda_2(t)$$

$$\gamma^{(n)}_{xx} = h_{n-1}\phi^{(2n)} + p_{n-1}\frac{d^{2(n-1)}}{dt^{2(n-1)}}\Lambda_3(t)$$

$$w^{(n)}(t, 0) = w^{(n)}_x(t, 0) \equiv 0$$

$$w^{(n)}_{xx}(t, 1) = -\lambda_1 f'_{n-1}(1)\frac{d^{2(n-1)}}{dt^{2(n-1)}}\Lambda_1(t) \tag{4.6}$$

$$z^{(n)}_{xx}(t, 1) = -\lambda_2 g'_{n-1}(1)\frac{d^{2(n-1)}}{dt^{2(n-1)}}\Lambda_2(t)$$

$$w^{(n)}_{xxx}(t, 1) = [\chi_1\alpha f_{n-1}(1) + f'_{n-1}(1)]\frac{d^{2(n-1)}}{dt^{2(n-1)}}\Lambda_1(t)$$

$$z_{xxx}^{(n)}(t, 1) = [\chi_2 \alpha g_{n-1}(1) + g'_{n-1}(1)] \frac{d^{2(n-1)}}{dt^{2(n-1)}} \Lambda_2(t)$$

$$\gamma^{(n)}(t, 0) = \gamma_x^{(n)}(t, 1) = 0$$

where $f_i(x)$, $g_i(x)$, $h_i(x)$ and $p_i(x)$ are polynomials.

It is necessary to choose the function $\dot{\phi}^{(0)}(t)$, which satisfies condition (2.8), in order to ensure that conditions (3.3) are satisfied up to the nth order inclusive. As in the previous problem the coefficients of the expansions (4.1) and (4.2) have the form of polynomials in x.

5. RESULTS OF CALCULATIONS

Calculations were performed for the following parameters:

$\varrho = 5500$ kg/m^3, $E = 1.5 \times 10^{11}$ N/m^2, $l = 1$ m, $T = 2$ s, $m = 14$ kg, $R_1 = 0.15$ m, $R_2 = R_3 = 0.1$ m, $\tau = 0.02$ s, $\tau_1 = 0.02$ s, $T_1 = 0.2$ s, $\eta_1 = 10$, $A = B = 0.05$ kg m^2, $C = 0.1$ kg m^2, $\mu = 9.167 \times 10^{-5}$, $\alpha = 0.507$, $\eta = 100$.

It was assumed that the cross-section of the beam is a circle of radius $r = 0.04$ m. The changes in velocity were taken in the form

$$\dot{\varphi}^{(0)}(t) = A_1 t^5 (t-\eta)^5, \quad \dot{\phi}^{(0)}(t) = A_2 t^5 (t-\eta_1)^5, \quad \tilde{z}_t(t) = A_3 \sin^5\left(\frac{\pi}{\eta} t\right)$$

It was also assumed that

$$\varphi(\eta) = -10.8, \quad \phi(0) = 0, \quad \phi(\eta_1) = -5\pi/4, \quad \tilde{z}(\eta) = 0.679$$

Hence

$$A_1 = -3 \times 10^{-18}, \quad A_2 = -10.9 \times 10^{-8}, \quad A_3 = 0.02$$

This corresponds to the maximum of the moduli of the angular velocities

$$\dot{\varphi}_{\max} \approx 14.6 \text{ s}^{-1}, \quad \dot{\phi}_{\max} \approx 53.3 \text{ s}^{-1}$$

Expansions up to the second order in μ were used in the calculations. Graphs of the control moments and forces for the model of the motion of the elastic manipulator with a useful load are shown in Fig. 3 (here the inverse transition from dimensionless quantities to dimensional ones was carried out).

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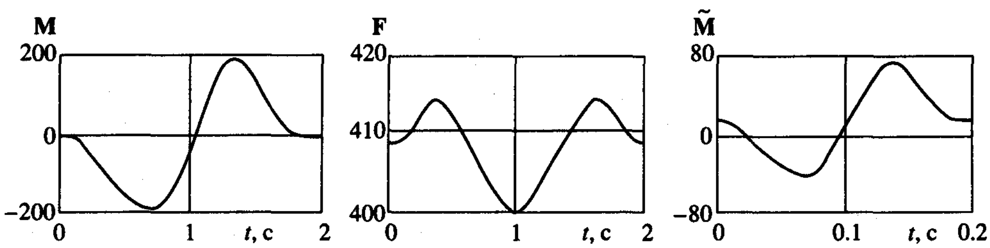

Fig. 3

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